ON GEOMETRIC PROBLEMS RELATED TO BROWN-YORK AND LIU-YAU QUASILOCAL MASS

PENGZI MIAO[1], YUGUANG SHI[2] AND LUEN-FAI TAM[3]

Abstract. We discuss some geometric problems related to the definitions of quasilocal mass proposed by Brown-York [5] [6] and Liu-Yau [13] [14]. Our discussion consists of three parts. In the first part, we propose a new variational problem on compact manifolds with boundary, which is motivated by the study of Brown-York mass. We prove that critical points of this variation problem are exactly static metrics. In the second part, we derive a derivative formula for the Brown-York mass of a smooth family of closed 2 dimensional surfaces evolving in an ambient three dimensional manifold. As an interesting by-product, we are able to write the ADM mass [1] of an asymptotically flat 3-manifold as the sum of the Brown-York mass of a coordinate sphere S_r and an integral of the scalar curvature plus a geometrically constructed function $\Phi(x)$ in the asymptotic region outside S_r. In the third part, we prove that for any closed, spacelike, 2-surface Σ in the Minkowski space $\mathbb{R}^{3,1}$ for which the Liu-Yau mass is defined, if Σ bounds a compact spacelike hypersurface in $\mathbb{R}^{3,1}$, then the Liu-Yau mass of Σ is strictly positive unless Σ lies on a hyperplane. We also show that the examples given by Ó Murchadha, Szabados and Tod [18] are special cases of this result.

1. Introduction

In this work, we will discuss some geometric problems related to the definitions of quasilocal mass proposed by Brown-York [5] [6] and Liu-Yau [13] [14]. In general, there are certain properties that a reasonable definition of quasilocal mass should satisfy, see [21] for example. The

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most important property is the positivity. There are results on positivity of Brown-York mass and Liu-Yau mass in [19, 14, 22, 20, 23]. In particular, the following is a consequence on the positivity of Brown-York mass proved by the last two authors in [19]. Let g_e be the standard Euclidean metric on $\mathbb{R}^3$. Let Ω be a bounded strictly convex domain in $\mathbb{R}^3$ with smooth boundary Σ which has mean curvature H_0. Then $\int_\Sigma H_0 d\sigma$ is a maximum of the functional $\int_\Sigma H d\sigma$ on the class of smooth metrics with nonnegative scalar curvature on Ω which agree with g_e tangentially on Σ and have positive boundary mean curvature H. It is interesting to see if this is still true for general domains in $\mathbb{R}^3$.

In [20], a similar result was proved for domains in $\mathbb{H}^3$, the hyperbolic 3-space. Namely, it was proved that if g_h is the standard hyperbolic metric on $\mathbb{H}^3$ and Ω is a bounded domain with strictly convex smooth boundary Σ which is a topological sphere and has mean curvature H_0, then $\int_\Sigma H_0 \cosh r d\sigma$ is a maximum of the functional $\int_\Sigma H \cosh r d\sigma$ on the class of smooth metrics with scalar curvature bounded below by -6 which agree with g_h tangentially on Σ and have positive boundary mean curvature H. Here r is the distance function on $\mathbb{H}^3$ from a fixed point in Ω. Again it is interesting to see if this is still true for general domains in $\mathbb{H}^3$.

The results and questions above motivate us to study the functional

$$F_\phi(g) = \int_\Sigma H\phi \, d\sigma,$$

where Σ is the boundary of an n dimensional compact manifold Ω, ϕ is a given smooth nontrivial function (that is $\phi \not\equiv 0$) on Σ, and $d\sigma$ is the volume form of a fixed metric γ on Σ. The class of metrics g we are interested is the space of metrics with constant scalar curvature K which induce the metric γ on Σ. In Theorem 2.1, we will prove the following: *g is a critical point of $F_\phi(\cdot)$ if and only if g is a static metric with a static potential N that equals ϕ on Σ. That is to say*:

$$\begin{cases} -(\Delta_g N)g + \nabla_g^2 N - N\mathrm{Ric}(g) &= 0, \quad \text{on } \Omega \\ N &= \phi, \quad \text{at } \Sigma. \end{cases}$$

In the theorem, for $K > 0$, we also assume that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive.

In particular, if $\phi = 1$, $K = 0$ and $n = 3$, we can conclude that g is a critical point of $\int_\Sigma H \, d\sigma$ if and only if g is a flat metric.

Another important question on quasilocal mass is whether it has some monotonicity property. In [19], it was shown that the Brown-York mass of the boundaries of certain domains in a space with some qausi-spherical metric is monotonically decreasing rather than increasing as

the domains become larger. In Theorem 3.1, we will derive *a more general formula for the derivative of the Brown-York mass of a smooth family of surfaces with positive Gaussian curvature which evolve in an ambient manifold*. The formula gives a generalization of the monotonicity formula in [19] which plays a key role in the proof of the positivity of Brown-York mass. As an interesting by-product of this derivative formula, in Corollary 3.5 we are able to write the ADM mass [1] of an asymptotically flat 3-manifold as the sum of the Brown-York mass of a coordinate sphere S_r and an integral of the scalar curvature plus a geometrically constructed function $\Phi(x)$ in the asymptotic region outside S_r.

The Minkowski space $\mathbb{R}^{3,1}$ represents the zero energy state in general relativity. Thus, a reasonable notion of quasilocal mass should be such that its value of a spacelike 2-surface in $\mathbb{R}^{3,1}$ equals zero. In [13, 14], the Liu-Yau mass was introduced and its positivity was proved. In the time symmetric case, this coincides with the Brown-York mass. However, Ó. Murchadha, Szabados and Tod [18] constructed spacelike 2-surfaces Σ with spacelike mean curvature vector $\vec{H}$ in $\mathbb{R}^{3,1}$ and with positive Gaussian curvature such that the Liu-Yau mass of Σ given by

$$\mathfrak{m}_{\text{LY}}(\Sigma) = \frac{1}{8\pi}\int_{\Sigma}(H_0 - |\vec{H}|)d\sigma$$

is strictly positive. Here H_0 is the mean curvature of Σ when isometrically embedded in $\mathbb{R}^3$ and $|\vec{H}|$ is the Lorentzian norm of $\vec{H}$ in $\mathbb{R}^{3,1}$. Recently, Wang and Yau [22, 23] introduce another definition of quasilocal mass to address this question. In Theorem 4.1 in this paper, we will prove the following: *Let Σ be a closed, connected, spacelike 2-surface in the Minkowski space $\mathbb{R}^{3,1}$ with spacelike mean curvature vector and with positive Gaussian curvature. Suppose Σ spans a compact, spacelike hypersurface in $\mathbb{R}^{3,1}$, then the Liu-Yau mass of Σ is strictly positive, unless Σ lies on a hyperplane.* The results give some properties on isometric embeddings of compact surfaces with positive Gaussian curvature in the Minkowski space. We will also show that all the examples in [18] satisfy the conditions in Theorem 4.1.

This paper is organized as follows. In section 2, we will prove that static metrics are the only critical points of the functional $F_\phi(\cdot)$. In section 3, a formula for the derivative of the Brown-York mass will be derived and some applications will be given. In section 4, we will prove that for "most" spacelike 2-surfaces in $\mathbb{R}^{3,1}$ for which the Liu-Yau mass is defined, their Liu-Yau mass is strictly positive. In the appendix, we prove some results on the differentiability of a 1-parameter family of

isometric embeddings in $\mathbb{R}^3$, following the arguments of Nirenberg [17]. The results will be used in section 3.

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2. static metrics and Brown-York type integral

Throughout this section, we let Ω be an n-dimensional ($n \geq 3$) compact manifold with smooth boundary Σ. Let γ be a smooth Riemannian metric on Σ. As in [16], for a constant K and any integer $k > \frac{n}{2} + 2$, we let $\mathcal{M}_\gamma^K$ be the set of $W^{k,2}$ metrics g on Ω with constant scalar curvature K such that $g|_{T(\Sigma)} = \gamma$. If $g \in \mathcal{M}_\gamma^K$ and the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive, where Δ_g is the usual Laplacian operator of g, then $\mathcal{M}_\gamma^K$ is a manifold near g (see [16] for detail). Let ϕ be a given smooth function on Σ, we define the following functional on $\mathcal{M}_\gamma^K$:

$$F_\phi(g) = \int_\Sigma H_g \phi \, d\sigma, \tag{2.1}$$

where H_g is the mean curvature of Σ in (Ω, g) with respect to the outward unit normal and $d\sigma$ is the volume form of γ. Motivated by the results in [19, 20] on the positivity of Brown-York mass and some generalization, we want to determine the critical points of $F_\phi(\cdot)$ on $\mathcal{M}_\gamma^K$.

Before we state the main result, we recall the following definition from [7]:

Definition 2.1. A metric g on an open set U is called a *static* metric on U if there exists a nontrivial function N (called the *static potential*) on U such that

$$-(\Delta_g N)g + \nabla_g^2 N - N\mathrm{Ric}(g) = 0. \tag{2.2}$$

Here Δ_g, ∇_g^2 are the usual Laplacian, Hessian operator of g and $\mathrm{Ric}(g)$ is the Ricci curvature of g.

A basic property of static metrics is that they are necessarily metrics of constant scalar curvature [7, Proposition 2.3].

In the following, we obtain a characterization of static metrics in $\mathcal{M}_\gamma^K$ using the function $F_\phi(\cdot)$.

Theorem 2.1. *With the above notations, let ϕ be a nontrivial smooth function on Σ. Suppose $g \in \mathcal{M}_\gamma^K$ such that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive. Then g is a critical point of $F_\phi(\cdot)$ defined in* (2.1) *if and only if g is a static metric with a static potential N such that $N = \phi$ on Σ.*

Proof. Since the Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive, we know $\mathcal{M}_\gamma^K$ is a manifold near g by the result in [16].

First, we suppose g is a static metric with a potential N such that $N = \phi$ on Σ. Let $g(t)$ be a smooth curve in $\mathcal{M}_\gamma^K$ with $g(0) = g$. Let

$$F(t) = \int_\Sigma H(t)\phi \, d\sigma,$$

where $H(t)$ is the mean curvature of Σ in $(\Omega, g(t))$ with respect to the outward unit normal ν. Let "$\,'\,$" denote the derivative with respect to t. We want to prove that $F'(0) = 0$. Let $h = g'(0)$. In what follows, we let ω_n denote the outward unit normal part of a 1-form ω, i.e. $\omega_n = \omega(\nu)$, let $\mathbb{II}$ be the second fundamental form of Σ in $(\Omega, g(t))$ with respect to ν, let X be the vector field on Σ that is dual to the 1-form $h(\nu,\cdot)|_{T(\Sigma)}$ on (Σ, γ) and let $\mathrm{div}_\gamma X$ be the divergence of X on (Σ,γ). For convenience, we often omit writing the volume form in an integral. As in [16, (34)], we have

(2.3)

$$\begin{aligned}
2F'(0) &= \int_\Sigma 2H'(0)N \\
&= \int_\Sigma N\left([d(\mathrm{tr}_g h) - \mathrm{div}_g h]_n - \mathrm{div}_\gamma X - \langle \mathbb{II}, h\rangle_\gamma\right) \\
&= \int_\Sigma N[d(\mathrm{tr}_g h) - \mathrm{div}_g h]_n - \int_\Sigma N \mathrm{div}_\gamma X \quad (\text{because } h|_{T\Sigma} = 0) \\
&= \int_\Omega N\left[\triangle_g(\mathrm{tr}_g h) - \mathrm{div}_g(\mathrm{div}_g h)\right] - \int_\Omega \mathrm{tr}_g h \Delta N + \int_\Sigma \mathrm{tr}_g h (dN)_n \\
&\quad - \int_\Omega \langle dN, \mathrm{div}_g h\rangle_g - \int_\Sigma N\mathrm{div}_\gamma X \\
&= -\int_\Omega N\langle h, \mathrm{Ric}(g)\rangle_g - \int_\Omega \mathrm{tr}_g h\Delta N + \int_\Sigma \mathrm{tr}_g h(dN)_n \\
&\quad + \int_\Omega \langle \nabla_g^2 N, h\rangle_g - \int_\Sigma h(\nu, \nabla N) - \int_\Sigma N\mathrm{div}_\gamma X \quad (\text{using (2.4) and (2.5)}) \\
&= \int_\Omega \langle h, -N\mathrm{Ric}(g) - (\Delta_g N)g + \nabla_g^2 N\rangle_g \\
&\quad + \int_\Sigma \mathrm{tr}_g h(dN)_n - \int_\Sigma h(\nu, \nabla N) - \int_\Sigma N\mathrm{div}_\gamma X,
\end{aligned}$$

where we have used the facts

$$\int_\Omega \langle dN, \mathrm{div}_g h\rangle_g = -\int_\Omega \langle \nabla_g^2 N, h\rangle_g + \int_\Sigma h(\nu, \nabla N) \tag{2.4}$$

and
(2.5)
$$DR_g(h) = \frac{d}{dt}R(t)|_{t=0} = -\triangle_g(\mathrm{tr}_g h) + \mathrm{div}_g(\mathrm{div}_g h) - \langle h, \mathrm{Ric}(g)\rangle_g = 0.$$
Here $R(t)$ is the scalar curvature of $g(t)$. Let $\nabla^\Sigma N$ denote the gradient of N on (Σ, σ). Integrating by parts on Σ, we have
(2.6)
$$\int_\Sigma N\mathrm{div}_\gamma X = -\int_\Sigma \langle \nabla^\Sigma N, X\rangle_\gamma = -\int_\Sigma h(\nu, \nabla N) + \int_\Sigma h(\nu,\nu)(dN)_n.$$
On the other hand, $\mathrm{tr}_g h = h(\nu,\nu)$ at Σ. Hence, $F'(0) = 0$ by (2.2), (2.3), and (2.6).

To prove the converse, suppose g is a critical point of $F_\phi(\cdot)$. Let $\{g(t)\}$ be any smooth path in $\mathcal{M}^K_\gamma$ passing $g = g(0)$. Let $h = g'(0)$ and $F(t) = F_\phi(g(t))$. As before, we have

$$\begin{aligned} 2F'(0) &= \int_\Sigma 2H'(0)\phi \\ &= \int_\Sigma \phi[d(\mathrm{tr}_g h) - \mathrm{div}_g h]_n - \int_\Sigma \phi \mathrm{div}_\gamma X. \end{aligned} \tag{2.7}$$

Since the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive and ϕ is not identically zero, there exists a unique smooth function $N = N_\phi$ which is not identically zero on Ω such that

$$\begin{cases} (n-1)\Delta_g N + KN &= 0, \quad on\ \Omega \\ N &= \phi, \quad at\ \Sigma. \end{cases} \tag{2.8}$$

With such an N given, we have

$$\begin{aligned} &\int_\Sigma \phi[d(\mathrm{tr}_g h) - \mathrm{div}_g h]_n - \int_\Sigma \phi\mathrm{div}_\gamma X \\ =& \int_\Omega N\left[\triangle_g(\mathrm{tr}_g h) - \mathrm{div}_g(\mathrm{div}_g h)\right] - \int_\Omega \mathrm{tr}_g h \Delta N + \int_\Sigma \mathrm{tr}_g h (dN)_n \\ &- \int_\Omega \langle dN, \mathrm{div}_g h\rangle_g - \int_\Sigma N\mathrm{div}_\gamma X \\ =& \int_\Omega N(-1)\langle h, \mathrm{Ric}(g)\rangle_g - \int_\Omega \mathrm{tr}_g h\Delta N + \int_\Omega \langle \nabla^2_g N, h\rangle_g, \end{aligned} \tag{2.9}$$

where we used the fact $DR_g(h) = 0$ (the boundary terms canceled as before and we have not used (2.8) yet).

Now let $\hat{h}$ be any smooth symmetric (0,2) tensor with compact support in Ω. For each t sufficiently small, we can find a smooth positive

function $u(t)$ on Ω such that $u(t) = 1$ at Σ and

$$g(t) = u(t)^{\frac{4}{n-2}}(g + t\hat{h}) \in \mathcal{M}_\gamma^K.$$

Moreover, $u(t)$ is differentiable at $t = 0$ and $u(0) \equiv 1$ on Ω. See the proof of [16, Theorem 5] for details on the existence of such a $u(t)$. Now $g'(0) = \frac{4}{n-2}u'(0)g + \hat{h}$. Hence, by (2.7) and (2.9) we have

$$\begin{aligned}
2F'(0) &= \int_\Omega \langle \frac{4}{n-2}u'(0)g + \hat{h}, -N\mathrm{Ric}(g) - (\Delta_g N)g + \nabla_g^2 N\rangle_g \\
&= \int_\Omega \langle \hat{h}, -N\mathrm{Ric}(g) - (\Delta_g N)g + \nabla_g^2 N\rangle_g \\
&\quad + \int_\Omega \frac{4}{n-2}u'(0)\left[-KN - n(\Delta_g N) + \Delta_g N\right].
\end{aligned} \tag{2.10}$$

By (2.8), the second integral in the above equation is zero. Hence, we have

$$2F'(0) = \int_\Omega \langle \hat{h}, -N\mathrm{Ric}(g) - (\Delta_g N)g + \nabla_g^2 N\rangle_g. \tag{2.11}$$

Since $\hat{h}$ can be arbitrary, we conclude that g and N satisfy (2.2). □

Remark 2.1. If $K \leq 0$, then the condition that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive holds automatically for $g \in \mathcal{M}_\gamma^K$.

As a direct corollary of Theorem 2.1, we have

Corollary 2.1. *With the notations given as in Theorem 2.1, suppose $K = 0$ and $\phi = 1$. Then $g \in \mathcal{M}_\gamma^0$ is a critical point of $\int_\Sigma H\, d\sigma$ if and only if g is a Ricci flat metric. In particular, if $n = 3$, then $g \in \mathcal{M}_\gamma^0$ is a critical point of $\int_\Sigma H\, d\sigma$ if and only if g is a flat metric.*

If ϕ does not change sign on the boundary, we further have:

Corollary 2.2. *With the notations given as in Theorem 2.1, suppose $\phi \geq 0$ or $\phi \leq 0$ on Σ. Suppose $g \in \mathcal{M}_\gamma^K$ is a static metric. If $K > 0$, we also assume that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive. Let $g(t)$ be a smooth family of smooth metrics on Ω with $g(0) = g$ such that*

(i) *the scalar curvature of $g(t)$ is at least K.*
(ii) *$g(t)$ induces γ on Σ.*

Then

$$\frac{d}{dt}F_\phi(g(t))|_{t=0} = 0.$$

Proof. We prove the case that $\phi \geq 0$ on Σ. The case that $\phi \leq 0$ on Σ is similar.

By the assumption of g, for t small, we can find smooth positive functions $u(t)$ on Ω with $u(t) = 1$ on Σ such that $\hat{g}(t) = u^{\frac{4}{n-2}}(t)g(t) \in \mathcal{M}^K_\gamma$, $u(t)$ is differentiable at $t = 0$ and $u(0) \equiv 1$ (see the proof of Proposition 1 in [16]). The mean curvature $\hat{H}(t)$ of Σ in $(\Omega, \hat{g}(t))$ is given by

$$\hat{H}(t) = H(t) + \frac{2(n-1)}{n-2}\frac{\partial u}{\partial \nu_t}, \tag{2.12}$$

where $H(t)$ and ν_t are the mean curvature and the unit outward normal of Σ in $(\Omega, g(t))$. Note that u satisfies:

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta_{g(t)}u - K(t)u & = -Ku^{\frac{n+2}{n-2}}, \text{ in } \Omega \\ u & = 1, \text{ on } \Sigma, \end{cases} \tag{2.13}$$

where $K(t)$ is the scalar curvature of $g(t)$. Since $K(t) \geq K$, by the maximum principle, we have

$$\frac{\partial u}{\partial \nu_t} \geq 0.$$

Hence, $\hat{H}(t) \geq H(t)$ and consequently $F_\phi(\hat{g}(t)) \geq F_\phi(g(t))$ by the assumption $\phi \geq 0$ on Σ. By Theorem 2.1, we have

$$\frac{d}{dt}F_\phi(\hat{g}(t))|_{t=0} = 0.$$

Since $\hat{g}(0) = g(0)$, we conclude

$$\frac{d}{dt}F_\phi(g(t))|_{t=0} = 0.$$

□

Here are some examples provided by Theorem 2.1:

Example 1: Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Σ. Then the standard Euclidean metric is a critical point of $F_\phi(\cdot)$ with $\phi \equiv 1$. If Σ is strictly convex, then this follows also from the result in [19].

Example 2: Let Ω be a bounded domain in $\mathbb{H}^n$ with smooth boundary Σ. Then the standard Hyperbolic metric is a critical point of $F_\phi(\cdot)$ with $\phi = \cosh r$, where r is the distance function on $\mathbb{H}^n$ from a fixed point. If Σ is strictly convex and $n = 3$, then this follows also from the result in [20].

Example 3: Let Ω be a domain in $\mathbb{S}^n$ with smooth boundary Σ such that the volume of Ω is less than 2π. Then the standard metric on $\mathbb{S}^n$ is

a critical point of $F_\phi(\cdot)$ with $\phi = \cos r$, where r is the distance function on $\mathbb{S}^n$ from a fixed point.

Example 4: Let Ω be a bounded domain with smooth boundary in the Schwarzschild manifold $\mathbb{R}^3 \setminus \{0\}$ with metric

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$$

with $m > 0$ and $r = |x|$. Then on Ω g is a critical point for $F_\phi(\cdot)$ with $\phi = (1 - \frac{m}{2r})/(1 + \frac{m}{2r})$.

Example 5: Complete conformally flat Riemannian manifolds with static metrics have been classified by Kobayashi in [12, Theorem 3.1]. In addition to the manifolds in the previous examples, there are other kind of static metrics with N being explicitly constructed. Domains in these manifolds will be critical points of $F_\phi(\cdot)$ where ϕ is the restriction of N to the boundary. See [12] for more details.

3. Derivative of the Brown-York mass

In this section, we give a derivative formula that describes how the Brown-York mass of a surface changes if the surface is evolving in an ambient Riemannian manifold. Our main result is:

Theorem 3.1. *Let $\mathbb{S}^2$ be the 2-dimensional sphere. Let (M, g) be a 3-dimensional Riemannian manifold. Let I be an open interval in $\mathbb{R}^1$. Suppose*

$$F : \mathbb{S}^2 \times I \longrightarrow M$$

is a smooth map such that, for $t \in I$,

(i) *$\Sigma_t = F(\mathbb{S}^2, t)$ is an embedded surface in M and Σ_t has positive Gaussian curvature.*

(ii) *The velocity vector $\frac{\partial F}{\partial t}$ is always perpendicular to Σ_t, i.e*

$$\frac{\partial F}{\partial t} = \eta \nu,$$

where ν is a given unit vector field normal to Σ_t and $\eta = \langle \frac{\partial F}{\partial t}, \nu \rangle$ denotes the speed of Σ_t with respect to ν.

Consider $\mathfrak{m}_{\text{BY}}(\Sigma_t)$, the Brown-York mass of Σ_t in (M, g), defined by

$$\mathfrak{m}_{\text{BY}}(\Sigma_t) = \frac{1}{8\pi} \int_{\Sigma_t} (H_0 - H)\, d\sigma_t, \tag{3.1}$$

where H_0 is the mean curvature of Σ_t with respect to the outward normal when isometrically embedded in $\mathbb{R}^3$, H is the mean curvature of Σ_t with respect to ν in (M, g), and $d\sigma_t$ is the volume form of the induced metric on Σ_t.

We have

$$\frac{d}{dt}\mathfrak{m}_{\text{BY}}(\Sigma_t) = \frac{1}{16\pi}\int_{\Sigma_t}\left(|A_0 - A|^2 - |H_0 - H|^2 + R\right)\eta \, d\sigma_t, \tag{3.2}$$

where A_0 is the second fundamental form of Σ_t with respect to the outward normal when isometrically embedded in $\mathbb{R}^3$, A is the second fundamental form of Σ_t with respect to ν in (M,g), and R is the scalar curvature of (M,g).

Our proof of Theorem 3.1 makes use of a recent formula of Wang and Yau (Proposition 6.1 in [23]):

Proposition 3.1. *Let Σ be an orientable closed embedded hypersurface in $\mathbb{R}^{n+1}$. Let $\{\Sigma_t\}_{|t|<\delta}$ be a smooth variation of Σ in $\mathbb{R}^{n+1}$. Then*

$$\frac{d}{dt}\left(\int_{\Sigma_t} H_0 \, d\sigma_t\right)|_{t=0} = \frac{1}{2}\int_{\Sigma}\left(H_0 \mathrm{tr}_\Sigma h - \langle A_0, h\rangle\right) d\sigma, \tag{3.3}$$

where H_0 and A_0 are the mean curvature and the second fundamental form of Σ with respect to the outward normal in $\mathbb{R}^{n+1}$, h is the variation of the induced metric σ on Σ, $\mathrm{tr}_\Sigma h = \langle \sigma, h\rangle$ denotes the trace of h with respect to σ, and $d\sigma_t$, $d\sigma$ denote the volume form on Σ_t, Σ.

In order to apply Proposition 3.1, we will need to show that, on a closed convex surface Σ in $\mathbb{R}^3$, an abstract metric variation on Σ indeed arises from a surface variation $\{\Sigma_t\}$ of Σ in $\mathbb{R}^3$. Precisely, we have:

Proposition 3.2. *Given an integer $k \geq 6$ and a number $0 < \alpha < 1$, let $\{\sigma(t)\}_{|t|<1}$ be a path of $C^{k,\alpha}$ metrics on $\mathbb{S}^2$ such that $\{\sigma(t)\}$ is differentiable at $t = 0$ in the space of $C^{k,\alpha}$ metrics. Suppose $\sigma(0)$ has positive Gaussian curvature. Then there exists a small number $\delta > 0$ and a path of $C^{k,\alpha}$ embeddings $\{f(t)\}_{|t|<\delta}$ of $\mathbb{S}^2$ in $\mathbb{R}^3$ such that $f(t)$ is an isometric embedding of $(\mathbb{S}^2, \sigma(t))$ for $|t| < \delta$ and $\{f(t)\}$ is differentiable at $t = 0$ in the space of $C^{2,\alpha}$ embeddings.*

The proposition above follows from the arguments by Nirenberg in [17]. For completeness, we include its proof here.

Proof. Given $\{\sigma(t)\}_{|t|<1}\}$, a path of $C^{k,\alpha}$ metrics on $\mathbb{S}^2$, let $h = \sigma'(0)$. Then h is a $C^{k,\alpha}$ symmetric (0,2) tensor. Since $\sigma(0)$ has positive Gaussian curvature, by the result in [17], there exists a $C^{k,\alpha}$ isometric embedding of $(\mathbb{S}^2, \sigma(0))$ in $\mathbb{R}^3$, which we denote by X. Given such an X, let $Y : \mathbb{S}^2 \to \mathbb{R}^3$ be a $C^{2,\alpha}$ solution to the linear equation

$$2dX \cdot dY = h, \tag{3.4}$$

where "$\cdot$" denotes the Euclidean dot product in $\mathbb{R}^3$ and (3.4) is understood as

$$dX(e_1) \cdot dY(e_2) + dX(e_2) \cdot dY(e_1) = h(e_1, e_2)$$

for any tangent vectors e_1, e_2 to $\mathbb{S}^2$. The existence of such a Y is provided by Theorem 2' in [17]. Let $d\bar{\sigma}^2 = h$ and let ϕ, p_1, p_2 be given as in (6.5), (6.6) in [17], then ϕ satisfies (6.15) in [17]. Using the fact that X is in $C^{k,\alpha}$ and $d\bar{\sigma}^2$ is in $C^{k,\alpha}$, we check that the coefficients of (6.15) in [17] (when written in a non-divergence form) is in $C^{k-3,\alpha}$. Thus, it follows from (6.15) in [17] that $\phi \in C^{k-1,\alpha}$, from which we conclude $Y \in C^{k-1,\alpha}$ by (6.11)-(6.13) in [17].

Now consider the $C^{k-1,\alpha}$ path of embeddings $\{G(t)\}_{|t|<t_0}$, where

$$G(t) = X + tY \tag{3.5}$$

and t_0 is chosen so that $G(t)$ is an embedding. Let g_e be the Euclidean metric on $\mathbb{R}^3$. The pull back metric $\tau(t) = G(t)^*(g_e)$ which is in $C^{k-2,\alpha}$ satisfies

$$\tau(0) = \sigma(0), \quad \tau'(0) = \sigma'(0), \tag{3.6}$$

which implies

$$||\tau(t) - \sigma(t)||_{C^{2,\alpha}} = O(t^2). \tag{3.7}$$

Apply Lemma 5.3 in the Appendix to $\sigma^0 = \sigma(0) = \tau(0)$, for each t sufficiently small, we can find a $C^{2,\alpha}$ isometric embedding $X(t)$ of $(\mathbb{S}^2, \sigma(t))$ in $\mathbb{R}^3$ such that

$$||G(t) - X(t)||_{C^{2,\alpha}} \le C||\tau(t) - \sigma(t)||_{C^{2,\alpha}} = O(t^2). \tag{3.8}$$

(By Lemma 1' in [17], $X(t)$ indeed lies in $C^{k,\alpha}$.) It follows from (3.8) that $\{X(t)\}$, when viewed as a path in the space of $C^{2,\alpha}$ embeddings, is differentiable at $t = 0$. Proposition 3.2 is therefore proved. □

Proposition 3.1 and Proposition 3.2 together imply:

Proposition 3.3. *Given an integer $k \ge 6$ and a number $0 < \alpha < 1$, suppose $\{\sigma(t)\}_{|t|<1}$ is a differentiable path in the space of $C^{k,\alpha}$ metrics on $\mathbb{S}^2$. Suppose $\sigma(t)$ has positive Gaussian curvature for each t. Let H_0 be the mean curvature of the isometric embedding of $(\mathbb{S}^2, \sigma(t))$ in $\mathbb{R}^3$ with respect to the outward normal. Let $d\sigma_t$ be the volume form of $\sigma(t)$. Then*

$$\int_{\mathbb{S}^2} H_0 \, d\sigma_t$$

is a differentiable function of t, and

$$\frac{d}{dt}\left(\int_{\mathbb{S}^2} H_0 \ d\sigma_t\right) = \frac{1}{2}\int_{\mathbb{S}^2} \langle H_0\sigma(t) - A_0, h\rangle\rangle \ d\sigma_t, \tag{3.9}$$

where A_0 is the second fundamental form of the isometric embedding of $(\mathbb{S}^2, \sigma(t))$ in $\mathbb{R}^3$ with respect to the outward normal, $h = \sigma'(t)$, "$\langle \cdot, \cdot \rangle$" denotes the metric product with respect to $\sigma(t)$ on the space of symmetric $(0,2)$ tensors.

Proof. Take any $t_0 \in (-1, 1)$. By Proposition 3.2, there exists a small positive number δ (depending on t_0) and a path of $C^{k,\alpha}$ embeddings $\{f(t)\}_{|t-t_0|<\delta}$ of $\mathbb{S}^2$ in $\mathbb{R}^3$, such that $f(t)$ is an isometric embedding of $(\mathbb{S}^2, \sigma(t))$ and $\{f(t)\}$ is differentiable at $t = t_0$ in the space of $C^{2,\alpha}$ embeddings.

Let $\Sigma_t = f(t)(\mathbb{S}^2)$, let $H_0(t)$ be the mean curvature of Σ_t with respect to the outward normal in $\mathbb{R}^3$, by definition we have

$$\int_{\mathbb{S}^2} H_0 \ d\sigma_t = \int_{\Sigma_t} H_0(t) \ d\sigma_t, \quad \forall \ |t - t_0| < \delta. \tag{3.10}$$

Apply the fact that $\{f(t)\}$ is differentiable at $t = t_0$ in the space of $C^{2,\alpha}$ embeddings and note that H_0 only involves derivatives of $f(t)$ up to the second order, we conclude that $\int_{\Sigma_t} H_0(t) \ d\sigma_t$ is differentiable at t_0. By (3.10), $\int_{\mathbb{S}^2} H_0 \ d\sigma_t$ is differentiable at t_0 as well. This shows $\int_{\mathbb{S}^2} H_0 \ d\sigma_t$ is a differentiable function of t. Equation (3.9) then follows directly from (3.10) and Proposition 3.1. □

We are now ready to prove Theorem 3.1 using Proposition 3.3.

Proof of Theorem 3.1. By Proposition 3.3, the function $\mathfrak{m}_{\text{BY}}(\Sigma_t)$ is a differentiable function of t. We have

$$\frac{d}{dt}\mathfrak{m}_{\text{BY}}(\Sigma_t) = \frac{1}{8\pi}\frac{d}{dt}\left(\int_{\Sigma_t} H_0 \ d\sigma_t\right) - \frac{1}{8\pi}\frac{d}{dt}\left(\int_{\Sigma_t} H \ d\sigma_t\right). \tag{3.11}$$

Let $\sigma = \sigma(t)$ be the induced metric on Σ_t. By (3.9) in Proposition 3.3, we have

$$\frac{d}{dt}\left(\int_{\Sigma_t} H_0 \ d\sigma_t\right) = \frac{1}{2}\int_{\Sigma_t} \langle H_0\sigma(t) - A_0, \frac{\partial \sigma}{\partial t}\rangle \ d\sigma_t. \tag{3.12}$$

Now, applying the fact that $\{\Sigma_t\}$ evolves in (M, g) according to

$$\frac{\partial F}{\partial t} = \eta\nu, \tag{3.13}$$

we have

$$\frac{\partial \sigma}{\partial t} = 2\eta A, \tag{3.14}$$

and

$$\frac{\partial H}{\partial t} = -\Delta\eta - (|A|^2 + \mathrm{Ric}(\nu,\nu))\eta, \tag{3.15}$$

where $\mathrm{Ric}(\nu,\nu)$ is the Ricci curvature of (M,g) along ν. Thus,

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Sigma_t} H_0 \, d\sigma_t\right) &= \frac{1}{2}\int_{\Sigma_t} \langle H_0\sigma(t) - A_0, 2\eta A\rangle \, d\sigma_t \\ &= \int_{\Sigma_t} H_0 H\eta - \langle A_0, A\rangle\eta \, d\sigma_t \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Sigma_t} H \, d\sigma_t\right) &= \int_{\Sigma_t} \frac{\partial H}{\partial t} + H^2\eta \, d\sigma_t \\ &= \int_{\Sigma_t} -(|A|^2 + \mathrm{Ric}(\nu,\nu))\eta + H^2\eta \, d\sigma_t. \end{aligned} \tag{3.17}$$

Hence, it follows from (3.11), (3.16) and (3.17) that

$$\frac{d}{dt}\mathfrak{m}_{\mathrm{BY}}(\Sigma_t) = \frac{1}{8\pi}\int_{\Sigma_t} \left[H_0H - \langle A_0, A\rangle + (|A|^2 + \mathrm{Ric}(\nu,\nu)) - H^2\right]\eta \, d\sigma_t. \tag{3.18}$$

Apply the Gauss equation to Σ_t in (M,g) and to the isometric embedding of Σ_t in $\mathbb{R}^3$ respectively, we have

$$2K = R - 2\mathrm{Ric}(\nu,\nu) + H^2 - |A|^2, \tag{3.19}$$

and

$$2K = (H_0)^2 - |A_0|^2, \tag{3.20}$$

where K is the Gaussian curvature of Σ_t. Hence, (3.18)-(3.20) imply that

$$\frac{d}{dt}\mathfrak{m}_{\mathrm{BY}}(\Sigma_t) = \frac{1}{16\pi}\int_{\Sigma_t} \left[|A_0 - A|^2 - (H_0 - H)^2 + R\right]\eta \, d\sigma_t. \tag{3.21}$$

Therefore, (3.2) is proved. $\square$

Next, we want to discuss some applications of Theorem 3.1. The first two applications below put the monotonicity property of the Brown-York mass in the construction in [19] into a more general context.

Corollary 3.1. *Let (M,g), I, F, $\{\Sigma_t\}$, η, A, A_0, H and H_0 be given as in Theorem 3.1 with $\eta > 0$. Suppose at each point $x \in \Sigma_t$, $t \in I$, $A_0 - A$ is either positive semi-definite or negative semi-definite, and $R \le 0$, then $\mathfrak{m}_{\mathrm{BY}}(\Sigma_t)$ is nonincreasing in t. If in addition, $A = \alpha A_0$*

for some number α depending on $x \in \Sigma_t$, then $\mathfrak{m}_{\text{BY}}(\Sigma_t)$ is constant in I if and only if $(\mathbb{S}^2 \times I, F^*(g))$ is a domain in $\mathbb{R}^3$.

Proof. Let λ_1, λ_2 be the eigenvalues of $A_0 - A$. Suppose $A_0 - A$ is either positive semi-definite or negative semi-definite, then $\lambda_1\lambda_2 \geq 0$ and hence $|A_0 - A|^2 - |H_0 - H|^2 = -2\lambda_1\lambda_2 \leq 0$. Since $R \leq 0$, by Theorem 3.1, we have:

$$\frac{d}{dt}\mathfrak{m}_{\text{BY}}(\Sigma_t) \leq 0$$

because $\eta > 0$. This proves the first assertion.

Suppose $(\mathbb{S}^2 \times I, F^*(g))$ is a domain in $\mathbb{R}^3$, then by definition we have $\mathfrak{m}_{\text{BY}}(\Sigma_t) = 0$, $\forall t$. Hence,

$$\frac{d}{dt}\mathfrak{m}_{\text{BY}}(\Sigma_t) = 0.$$

Conversely, suppose $A = \alpha A_0$ and

$$\frac{d}{dt}\mathfrak{m}_{\text{BY}}(\Sigma_t) = 0.$$

Then $R = 0$ and $A = A_0$. In particular, $H = H_0$. For any $(t_1, t_2) \subset I$, let $\Omega = \mathbb{S}^2 \times (t_1, t_2)$ with the pull back metric $F^*(g)$. Let D be the interior of $\Sigma_{t_1} = F(\mathbb{S}^2 \times \{t_1\})$ when it is isometrically embedded in $\mathbb{R}^3$ and E be the exterior of $\Sigma_{t_2} = F(\mathbb{S}^2 \times \{t_2\})$ when it is isometrically embedded in $\mathbb{R}^3$. By gluing Ω with D along $\mathbb{S}^2 \times \{t_1\}$, which is identified with Σ_{t_1} through F, and gluing Ω with E along $\mathbb{S}^2 \times \{t_2\}$, which is identified with Σ_{t_2} through F, we have an asymptotically flat and scalar flat manifold with corners and with zero mass, and it must be flat by [15] [19]. Hence, Ω is flat. Since it is simply connected, Ω can be isometrically embedded in $\mathbb{R}^3$.

□

Corollary 3.2. *Let (M, g), I, F, $\{\Sigma_t\}$, η, A, A_0, H and H_0 be given as in Theorem 3.1. Let g_e be the Euclidean metric on $\mathbb{R}^3$. Suppose there exists another smooth map*

$$F^0 : \mathbb{S}^2 \times I \longrightarrow \mathbb{R}^3$$

such that

(i) *$\Sigma_t^0 = F^0(\mathbb{S}^2, t)$ is an embedded closed convex surface in $\mathbb{R}^3$ and*

$$(F_t^0)^*(g_e) = F_t^*(g),$$

where $F_t^0(\cdot) = F^0(\cdot, t)$ and $F_t(\cdot) = F(\cdot, t)$.

(ii) *The velocity vector* $\frac{\partial F^0}{\partial t}$ *is always perpendicular to* Σ_t^0*, i.e*

$$\frac{\partial F^0}{\partial t} = \eta^0 \nu^0,$$

where ν^0 *is the outward unit normal to* Σ_t^0 *in* $\mathbb{R}^3$ *and* η^0 *denotes the speed of* Σ_t^0 *with respect to* ν^0*.*

Suppose $\eta^0 > 0$*,* $\eta > 0$ *and* (M, g) *has zero scalar curvature, then the Brown-York mass* $\mathfrak{m}_{\text{BY}}(\Sigma_t)$ *is monotonically non-increasing, and* $\mathfrak{m}_{\text{BY}}(\Sigma_t)$ *is a constant if and only if* $(\mathbb{S}^2 \times I, F^*(g))$ *is a domain in* $\mathbb{R}^3$*.*

Proof. Since $\eta^0 > 0$ and $\eta > 0$, we can write $(F^0)^*(g_e)$ and $F^*(g)$ as

$$F^*(g_e) = (\eta^0)^2 dt^2 + g_t \quad \text{and} \quad F^*(g) = \eta^2 dt^2 + g_t, \tag{3.22}$$

where g_t denotes the same induced metric on both Σ_t^0 and Σ_t. Now it follows from (3.22) that

$$A = \frac{\eta^0}{\eta} A_0. \tag{3.23}$$

Since A_0 is positive definite, the results follow from Corollary 3.1. $\square$

Remark 3.1. We note that

(i) Quasi-spherical metrics constructed in [19] satisfy all the assumptions of Corollary 3.2.
(ii) In case $\eta^0 = 1$, one recovers the monotonicity formula in [19].

By applying the co-area formula directly to (3.2), we also obtain

Corollary 3.3. *Let* (M, g)*,* F*,* $\{\Sigma_t\}$*,* η*,* A*,* A_0*,* H *and* H_0 *be given as in Theorem 3.1. Suppose* $\eta > 0$*. For any* $t_1 < t_2$*, let* $\Omega_{[t_1,t_2]}$ *be the region bounded by* Σ_{t_1} *and* Σ_{t_2}*. Then*

$$\mathfrak{m}_{\text{BY}}(\Sigma_{t_2}) - \mathfrak{m}_{\text{BY}}(\Sigma_{t_1}) = \frac{1}{16\pi}\left(\int_{\Omega_{[t_1,t_2]}} R \, dV + \int_{\Omega_{[t_1,t_2]}} \Phi \, dV\right), \tag{3.24}$$

where R *is the scalar curvature of* (M, g)*,* dV *is the volume form of* g *on* M*, and* Φ *is the function on* $\Omega_{[t_1,t_2]}$*, depending on* $\{\Sigma_t\}$*, defined by*

$$\Phi(x) = |A_0 - A|^2 - (H_0 - H)^2, \ x \in \Sigma_t. \tag{3.25}$$

The function $\Phi(x)$ defined above clearly depends on the foliation $\{\Sigma_t\}$ connecting Σ_{t_1} to Σ_{t_2}. However, it is interesting to note that the integral $\int_{\Omega_{[t_1,t_2]}} \Phi \, dV$ turns out to be $\{\Sigma_t\}$ independent by (3.24).

We can apply formula (3.24) to small geodesic balls in a general 3-manifold and to asymptotically flat regions in an asymptotically flat 3-manifold.

Corollary 3.4. *Let (M,g) be a 3-dimensional Riemannian manifold. Let $p \in M$ and $B_\delta(p)$ be a geodesic ball centered at p with geodesic radius δ. Suppose δ is small enough such that*

(1) *$\delta < i_p(M)$, where $i_p(M)$ is the injectivity radius of (M,g) at p.*
(2) *For any $0 < r \le \delta$, the geodesic sphere $S_r(p)$, centered at p with geodesic radius r, has positive Gaussian curvature.*

Then the Brow-York mass of $S_\delta(p)$ can be written as

$$\mathfrak{m}_{\mathrm{BY}}(S_\delta(p)) = \frac{1}{16\pi}\left(\int_{B_\delta(p)} R\, dV + \int_{B_\delta(p)\setminus\{p\}} \Phi\, dV\right), \tag{3.26}$$

where R is the scalar curvature of M, dV is the volume form on M, and Φ is the function on $B_\delta(p)\setminus\{p\}$, defined by

$$\Phi(x) = |A_0 - A|^2 - (H_0 - H)^2,\ x \in S_r. \tag{3.27}$$

Here A, H are the second fundamental form, the mean curvature of S_r in M with respect to the outward normal; and A_0, H_0 are the second fundamental form, the mean curvature of the isometric embedding of S_r in $\mathbb{R}^3$ with respect to the outward normal.

Proof. Let (r,ω) be the geodesic polar coordinate of $x \in B_\delta(p)\setminus\{p\}$, where r denotes the distance from x to p. Since $\frac{\partial}{\partial r} \perp S_r$, we can choose the foliation $\{\Sigma_t\}$ in Corollary 3.1 to be $\{S_r\}$ with $t = r$. By (3.24), we have

$$\mathfrak{m}_{\mathrm{BY}}(S_\delta(p)) - \mathfrak{m}_{\mathrm{BY}}(S_r(p)) = \frac{1}{16\pi}\int_{B_\delta(p)\setminus B_r(p)} (R+\Phi)\, dV. \tag{3.28}$$

By [8], we have

$$\lim_{r\to 0+} \mathfrak{m}_{\mathrm{BY}}(S_r(p)) = 0. \tag{3.29}$$

Hence, (3.26) follows from (3.28) and (3.29). □

Next, we express the ADM mass [1] as the sum of the Brown-York mass of a coordinate sphere and an integral involving the scalar curvature and the function $\Phi(x)$.

Corollary 3.5. *Let (M,g) be an asymptotically flat 3-manifold with a given end. Let $\{x^i \mid i = 1,2,3\}$ be a coordinate system at ∞ defining the asymptotic structure of (M,g). Let $S_r = \{x \in M \mid |x| = r\}$ be the coordinate sphere, where $|x|$ denotes the coordinate length. Suppose $r_0 \gg 1$ is a constant such that S_r has positive Gaussian curvature for each $r \ge r_0$. Then*

$$\mathfrak{m}_{\mathrm{ADM}} = \mathfrak{m}_{\mathrm{BY}}(S_{r_0}) + \frac{1}{16\pi}\int_{M\setminus D_{r_0}} R\, dV + \frac{1}{16\pi}\int_{M\setminus D_{r_0}} \Phi\, dV, \tag{3.30}$$

where $\mathfrak{m}_{\text{ADM}}$ is the ADM mass of (M,g), R is the scalar curvature of (M,g), D_{r_0} is the bounded open set in M enclosed by S_{r_0}, and Φ is the function on $M \setminus D_{r_0}$ defined by

$$\Phi(x) = |A_0 - A|^2 - (H_0 - H)^2, \ x \in S_r. \tag{3.31}$$

Here A, H are the second fundamental form, the mean curvature of S_r in M; and A_0, H_0 are the second fundamental form, the mean curvature of S_r when isometrically embedded in $\mathbb{R}^3$.

Proof. $\{S_r\}_{r \geq r_0}$ consists of level sets of the function r on $M \setminus D_{r_0}$, hence can be *reparameterized* to evolve in a way that its velocity vector is perpendicular to the surface at each time. To be precise, we can define the vector field $X = \frac{\nabla r}{|\nabla r|^2}$ on $M \setminus D_{r_0}$ and let $\gamma_p(t)$ be the integral curve of X starting at $p \in S_{r_0}$. For any $t \geq 0$, let $\Sigma_t = \{\gamma_p(t) \mid p \in S_{r_0}\}$, then $\Sigma_t = S_{r_0+t}$. For any $T > 0$, apply (3.24) to $\{\Sigma_t\}_{0 \leq t \leq T}$, we have

$$\mathfrak{m}_{\text{BY}}(S_{r_0+T}) - \mathfrak{m}_{\text{BY}}(S_{r_0}) = \frac{1}{16\pi}\left(\int_{\Omega_{[0,T]}} R \, dV + \int_{\Omega_{[0,T]}} \Phi \, dV\right), \tag{3.32}$$

where $\Omega_{[0,T]}$ is the region in M bounded by $\Sigma_0 = S_{r_0}$ and $\Sigma_T = S_{r_0+T}$. Letting $T \to +\infty$, by [8] we have

$$\lim_{T \to +\infty} \mathfrak{m}_{\text{BY}}(S_T) = \mathfrak{m}_{\text{ADM}}. \tag{3.33}$$

Hence, (3.30) follows from (3.32) and (3.33). □

4. Liu-Yau mass of spacelike two-surfaces in $\mathbb{R}^{3,1}$

Let Σ be a closed, connected, 2-dimensional spacelike surface in a spacetime N. Suppose Σ has positive Gaussian curvature and has spacelike mean curvature vector $\vec{H}$ in N. Let H_0 be the mean curvature of Σ with respect to the outward unit normal when it is isometrically embedded in $\mathbb{R}^3$. The Liu-Yau mass of Σ is then defined as (see [13, 14]):

$$\mathfrak{m}_{\text{LY}}(\Sigma) = \frac{1}{8\pi}\int_\Sigma (H_0 - |\vec{H}|) \, d\sigma,$$

where $|\vec{H}|$ is Lorentzian norm of $\vec{H}$ in N and $d\sigma$ is the volume form of the induced metric on Σ.

In [14], the following positivity result was proved: *Let Ω be a compact, spacelike hypersurface in a spacetime N satisfying the dominant energy conditions. Suppose the boundary $\partial\Omega$ has finitely many components Σ_i, $1 \leq i \leq l$, each of which has positive Gaussian curvature and has spacelike mean curvature vector in N. Then $\mathfrak{m}_{\text{LY}}(\Sigma_i) \geq 0$ for all i;*

moreover if $\mathfrak{m}_{\text{LY}}(\Sigma_i) = 0$ *for some* i*, then* $\partial\Omega$ *is connected and* N *is a flat spacetime along* Ω*.*

We remark that in their proof of the above result, it is assumed implicitly that the mean curvature of $\partial\Omega$ in Ω with respect to the outward unit normal is positive. See the statement [22, Theorem 1.1]. The condition is necessary as can be seen by the following example in the time symmetric case:

Let g_e be the Euclidean metric on $\mathbb{R}^3$ and let $m > 0$ be a constant. Consider the Schwarzschild metric (with negative mass)

$$g = \left(1 - \frac{m}{2|x|}\right)^4 g_e,$$

defined on $\{0 < |x| < \frac{m}{2}\}$. Given any $0 < r_1 < r_2 < \frac{m}{2}$, consider the domain

$$\Omega = \{r_1 < |x| < r_2\}.$$

For any constant r, the mean curvature H of the sphere $S_r = \{|x| = r\}$ with respect to the unit normal in the direction of $\partial/\partial r$ is

$$H = \frac{1}{(1 - \frac{m}{2r})^2}\left(\frac{2}{r} + \frac{4}{1 - \frac{m}{2r}}\frac{m}{2r^2}\right).$$

The mean curvature of S_r when it is embedded in $\mathbb{R}^3$ is

$$H_0 = \frac{1}{(1 - \frac{m}{2r})^2}\frac{2}{r}.$$

Suppose $r < \frac{m}{2}$, then $H < 0$ and

$$|H| - H_0 = -\frac{4}{r(1 - \frac{m}{2r})^3} > 0.$$

Hence, $\mathfrak{m}_{\text{LY}}(S_{r_1}) < 0$ and $\mathfrak{m}_{\text{LY}}(S_{r_2}) < 0$ where $\partial\Omega = S_{r_1} \cup S_{r_2}$.

In [18], Ó Murchadha, Szabados and Tod gave some examples of a spacelike 2-surface, lying on the light cone of the Minkowski space $\mathbb{R}^{3,1}$, whose Liu-Yau mass is strictly positive. Motivated by their result, we want to understand the Liu-Yau mass of more general spacelike 2-surfaces in $\mathbb{R}^{3,1}$. In the sequel, we always regard $\mathbb{R}^3$ as the $t = 0$ slice in $\mathbb{R}^{3,1}$. We have the following:

Theorem 4.1. *Let* Σ *be a closed, connected, smooth, spacelike* 2*-surface in* $\mathbb{R}^{3,1}$*. Suppose* Σ *spans a compact spacelike hypersurface in* $\mathbb{R}^{3,1}$*. If* Σ *has positive Gaussian curvature and has spacelike mean curvature vector, then* $\mathfrak{m}_{\text{LY}}(\Sigma) \geq 0$*; moreover* $\mathfrak{m}_{\text{LY}}(\Sigma) = 0$ *if and only if* Σ *lies on a hyperplane in* $\mathbb{R}^{3,1}$*.*

In order to prove this theorem, we need the following result which can be proved by the method of Bartnik and Simon [4] and by an idea from Bartnik [2]. In fact, it is just a special case of the results by Bartnik [3].

Lemma 4.1. *Let Σ be a closed, connected, smooth, spacelike 2-surface in $\mathbb{R}^{3,1}$. Suppose Σ spans a compact spacelike hypersurface in $\mathbb{R}^{3,1}$. Then Σ spans a compact, smoothly immersed, maximal spacelike hypersurface in $\mathbb{R}^{3,1}$.*

Proof. Let M be a compact spacelike hypersurface in $\mathbb{R}^{3,1}$ spanned by Σ. By extending M a bit, we may assume that there exists a spacelike hypersurface $\tilde{M}$ in $\mathbb{R}^{3,1}$ such that $\overline{M} \subset \tilde{M}$. Since $\tilde{M}$ is spacelike, $\tilde{M}$ is *locally* a graph over an open set in $\mathbb{R}^3$. Hence, the projection map $\pi : \tilde{M} \to \mathbb{R}^3$, given by $\pi(x,t) = x$, is a local diffeomorphism. Now consider the map

$$F : \tilde{M} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^{3,1}, \tag{4.1}$$

given by $F(p,s) = (x,s)$ for any $p = (x,t) \in \tilde{M}$, then F is a local diffeomorphism as well. Let $N = \tilde{M} \times \mathbb{R}^1$ equipped with the pull back metric. Let v be the time function on $\tilde{M}$ in $\mathbb{R}^{3,1}$, i.e. $v(x,t) = t$. Since v is a smooth function on $\tilde{M}$, we can consider its graph in N. Let $\hat{\Sigma}$ and $\hat{G}$ be the graph of v over Σ and $\overline{M}$ in N respectively. Then $\hat{G}$ is a compact, spacelike hypersurface in N whose boundary is $\hat{\Sigma}$. Moreover, $F|_{\hat{G}} : \hat{G} \to \overline{M} \subset \mathbb{R}^{3,1}$ and $F|_{\hat{\Sigma}} : \hat{\Sigma} \to \Sigma \subset \mathbb{R}^{3,1}$ are both isometries.

Now one can carry over the arguments in section 3 in [4] to prove that there is a smooth solution (defined on M) to the maximal surface equation in N such that, if G is its graph in N, then $\partial G = \hat{\Sigma}$. For example, Lemma 3.3 in [4] can be rephrased as: For $\theta > 0$, let

$$\mathcal{D} = \{\phi \in C^{0,1}(\overline{M}) |\ |D\phi| \le (1-\theta)\}$$

and

$$\begin{aligned}\mathcal{F} = \{u \in C^2(M) \mid |Du| < 1 \text{ with maximal graph} \\ \text{and } u = \phi \text{ on } \Sigma \text{ for some } \phi \in \mathcal{D}\}.\end{aligned} \tag{4.2}$$

Then there exists $r_0 >$ and $\theta_1 > 0$ such that for all $u \in \mathcal{F}$ and for all $p, q \in M$ with $d(p,\Sigma), d(q,\Sigma) < \frac{1}{3} r_0$(say) and $d(p,q) = r_0$ we have: $|u(p) - u(q)| \le (1-\theta_1) r_0$.

One then readily checks that $F(G)$ is a compact, smoothly immersed, maximal hypersurface in $\mathbb{R}^{3,1}$ spanned by $\Sigma = F(\hat{\Sigma})$. $\square$

To prove Theorem 4.1, we also need a technical lemma concerning the boundary mean curvature of a compact spacelike hypersurface in $\mathbb{R}^{3,1}$, whose boundary has spacelike mean curvature vector.

Lemma 4.2. *Let M be a compact 3-manifold with boundary ∂M. Let $F : M \to \mathbb{R}^{3,1}$ be a smooth, maximal spacelike immersion such that $F|_{\partial M} : \partial M \to F(\partial M) \subset \mathbb{R}^{3,1}$ has spacelike mean curvature vector. Let g be the pull back metric on M and k be the mean curvature of ∂M in (M, g) with respect to the outward unit normal. Then k must be nonnegative at some point on ∂M.*

Proof. Suppose $k < 0$ everywhere on ∂M. Since $F(M)$ is a compact subset in $\mathbb{R}^{3,1}$, without loss of generality, we may assume that $F(M) \subset \{x_1 \leq 0\}$ and $F(M) \cap \{x_1 = 0\} \neq \emptyset$. Let $X_0 = F(q) \in F(M) \cap \{x_1 = 0\}$ for some $q \in M$. If q is an interior point of M, then there exists an open neighborhood V of q in the interior of M such that the tangent space of $F(V)$ at X_0 is $\{x_1 = 0\}$. This is impossible, because $F(V)$ needs to be spacelike. Therefore, $q \in \partial M$. Using the fact that F is a spacelike immersion again, we know there exists an open neighborhood U of q in M such that $F(\overline{U})$ is a spacelike graph of some function f over $\overline{D}$ for some open set $D \subset \mathbb{R}^3 \cap \{x_1 \leq 0\}$. Let $B = F(\overline{U} \cap \partial M)$ and let $\hat{B}$ be the part of ∂D such that B is the graph of f over $\hat{B}$. We note that $X_0 \in B$. Without loss of generality, we may assume that X_0 is the origin.

To proceed, we let $T = \frac{\partial}{\partial t}$ and define the following notations:

n: the future time like unit normal to $F(\overline{U})$ in $\mathbb{R}^{3,1}$;

ν: the unit outward normal to B in $F(\overline{U})$;

$\hat{\nu}$: the unit outward normal to $\hat{B}$ in $\overline{D}$.

We parallel translate ν, $\hat{\nu}$ and all the tangent vectors of B, $\hat{B}$ along the T direction. Also, we consider f as a function on $D \times (-\infty, \infty)$ so that f is independent of t.

Now $\hat{\nu}$ is normal to B, so $\hat{\nu} = u\nu + vn$ for some numbers u, v satisfying $u^2 - v^2 = 1$. At $X_0 \in B$, we have $\hat{\nu} = \frac{\partial}{\partial x_1}$. Suppose $\alpha(s) = (x_1(s), x_2(s), x_3(s), t(s))$ is a curve in $F(\overline{U})$ such that $\alpha(0) = X_0$ and $\alpha'(0) = \nu$. Then, for $t < 0$ small, $\alpha(t) \in F(\overline{U})$ and so $x_1(t) < 0$. Since $x_1(0) = 0$, we have $x_1'(0) \geq 0$, hence $u = \langle \hat{\nu}, \nu \rangle \geq 0$. Since $u^2 = 1 + v^2$, we have $u > |v|$ at X_0.

Let $\vec{H}$ be the mean curvature vector of B in $\mathbb{R}^{3,1}$. Let $p = p_{ij}$ be the second fundamental form of $F(\overline{U})$ in $\mathbb{R}^{3,1}$ with respect to n. Then

$$\vec{H} = -k\nu + (\operatorname{tr}_B p)n, \tag{4.3}$$

where $\mathrm{tr}_B p$ denotes the trace of p restricted to B. Hence,

$$-\langle \vec{H}, \hat{\nu}\rangle = uk + v(\mathrm{tr}_B p). \tag{4.4}$$

At X_0, we have shown $u > |v|$. On the other hand, we know $|k| > |\mathrm{tr}_B p|$ (because $\vec{H}$ is spacelike) and $k < 0$ (by the assumption), therefore we have $-\langle \vec{H}, \hat{\nu}\rangle < 0$ at X_0. Recall that

$$\langle \vec{H}, \hat{\nu}\rangle = \langle \sum_{i=1}^{2} \nabla_{e_i} e_i, \nu\rangle, \tag{4.5}$$

where $\{e_1, e_2\}$ is an orthonormal frame in $T_{X_0}B$ and ∇ is the covariant derivative in $\mathbb{R}^{3,1}$. Hence there exists a unit vector $e \in T_{X_0}B$ such that

$$-\langle \nabla_e e, \hat{\nu}\rangle < 0. \tag{4.6}$$

Suppose e is the tangent of a curve $\gamma(s) \subset B$ at $s = 0$. Let $\hat{\gamma}(s) \subset \hat{B}$ be the projection of $\gamma(s)$ in $\mathbb{R}^3$. Then

$$\gamma'(s) = \hat{\gamma}'(s) + \frac{d}{ds} f(\hat{\gamma}(s))T. \tag{4.7}$$

Hence,

$$\begin{aligned} -\langle \nabla_{\gamma'(s)} \gamma'(s), \hat{\nu}\rangle &= -\langle \nabla_{\gamma'(s)} \hat{\gamma}'(s), \hat{\nu}\rangle \\ &= -\langle \nabla_{\hat{\gamma}'(s)} \hat{\gamma}'(s), \hat{\nu}\rangle, \end{aligned} \tag{4.8}$$

where we have used the facts that T is parallel, $T \perp \hat{\nu}$ and $\hat{\gamma}'(s)$ is parallel translated along T. Thus, it follows from (4.6), (4.8) and the fact $e = \gamma'(0)$ that

$$-\langle \nabla_{\hat{\gamma}'(0)} \hat{\gamma}'(0), \hat{\nu}\rangle < 0. \tag{4.9}$$

But this is impossible because $\hat{\gamma}(s) \subset \hat{B} \subset \{x_1 \le 0\} \cap \mathbb{R}^3$ and $\hat{\nu} = \frac{\partial}{\partial x_1}$ at $\hat{\gamma}(0)$. Therefore, we have proved that k can *not* be negative everywhere on ∂M. $\square$

Proof of Theorem 4.1. By Lemma 4.1, we know that Σ indeed bounds a compact, smoothly immersed, maximal spacelike hypersurface in $\mathbb{R}^{3,1}$. Precisely, this means that there exists a compact 3-manifold M with boundary ∂M and a smooth, maximal spacelike immersion $F : M \to \mathbb{R}^{3,1}$ such that $F : \partial M \to \Sigma$ is a diffeomorphism.

Let $g = g_{ij}$ be the pull back metric on M. Let $p = p_{ij}$ be the second fundamental form of the immersion $F : M \to \mathbb{R}^{3,1}$. Let R be the scalar curvature of (M, g). Since F is a maximal immersion, it follows from the constraint equations (or simply the Gauss equation) that

$$R = |p|^2 \ge 0, \tag{4.10}$$

where "$|\cdot|$" is taken with respect to g. On the other hand, let k be the mean curvature of ∂M in (M, g) with respect to the outward unit normal and let $\vec{H}$ be the mean curvature vector of $\Sigma = F(\partial M)$ in $\mathbb{R}^{3,1}$, it is known that

$$|\vec{H}|^2 = k^2 - (\mathrm{tr}_{\Sigma} p)^2, \tag{4.11}$$

where $\mathrm{tr}_{\Sigma} p$ is the trace of p restricted to Σ. Since $\vec{H}$ is spacelike, (4.11) implies that either $k > 0$ or $k < 0$ on ∂M because ∂M is connected. By Lemma 4.2, we have $k > 0$ on ∂M.

Now let k_0 be the mean curvature of Σ with respect to the unit outward normal when it is isometrically embedded in $\mathbb{R}^3$. It follows from (4.11) that

$$\int_{\Sigma} (k_0 - |\vec{H}|) \, d\sigma \geq \int_{\Sigma} (k_0 - k) \, d\sigma. \tag{4.12}$$

On the other hand, by the result of [19], we have

$$\int_{\Sigma} (k_0 - k) \, d\sigma \geq 0 \tag{4.13}$$

and equality holds if and only if (M, g) is a domain in $\mathbb{R}^3$. Thus, we conclude from (4.12) and (4.13) that $\mathfrak{m}_{\mathrm{LY}}(\Sigma) \geq 0$. Moreover, if $\mathfrak{m}_{\mathrm{LY}}(\Sigma) = 0$, then (M, g) must be flat, hence $R = 0$ and consequently $p = 0$. Therefore, $F(M)$ and hence Σ lie on a hyperplane in $\mathbb{R}^{3,1}$. Conversely, if Σ lies on a hyperplane in $\mathbb{R}^{3,1}$, then obviously $\mathfrak{m}_{\mathrm{LY}}(\Sigma) = 0$. $\square$

In the sequel, we want to show that the examples given in [18] satisfy the assumption in Theorem 4.1. To do that, we need the following definition:

Definition 4.1. Two points p and q in a Lorentzian manifold N are said to be *causally related* if p and q can be joined by a timelike or null path. A set S in N is called *acausal* if no two points in S are causally related.

We claim that *all surfaces in the examples in* [18] *are acausal.* Suppose this claim is true, then by Theorem 3 in [9](P.4765), we know that those surfaces span spacelike hypersurfaces in $\mathbb{R}^{3,1}$, hence satisfying the assumption in Theorem 4.1.

To verify the claim, let Σ be an example given in [18], i.e. in terms of the usual spherical coordinates (t, r, θ, ϕ) in $\mathbb{R}^{3,1}$, Σ is determined by the equation

$$t = r = F(\theta, \phi), \tag{4.14}$$

where $F = F(\theta, \phi)$ is a smooth positive function of $(\theta, \phi) \in \mathbb{S}^2$. Suppose Σ is *not acausal*, then there exists two distinct points p, q in Σ and a path $\gamma(\tau)$ in Σ such that $\gamma(0) = p$, $\gamma(1) = q$, and

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 \leq (\dot{t})^2, \quad \forall\, \tau \in [0,1], \tag{4.15}$$

here we denote $\gamma(\tau) = (x(\tau), y(\tau), z(\tau), t(\tau))$ and " $\cdot$ " denotes the derivative with respect to τ. Since $\dot{\gamma} \neq 0$, without loss of generality, we may assume $\dot{t} > 0$. Let $r = \sqrt{x^2 + y^2 + z^2}$, (4.15) implies that

$$|\dot{r}| \leq \dot{t}. \tag{4.16}$$

Note that $r(0) = t(0)$ and $r(1) = t(1)$, we see that

$$|\dot{r}| = \dot{t}, \tag{4.17}$$

for all $\tau \in [0,1]$. By the equality case in the Cauchy-Schwartz inequality, we must have

$$x = k(\tau)\dot{x}, \;\; y = k(\tau)\dot{y}, \;\; z = k(\tau)\dot{z} \tag{4.18}$$

for some function $k = k(\tau)$ and for all $\tau \in [0,1]$. Clearly, this implies that p and q lie on a line which passes through the origin, or equivalently, $p = aq$ for some positive number a. On the other hand, using the Cartesian coordinates, we may write p as

$$(F(\theta_1, \phi_1), F(\theta_1, \phi_1)\sin\theta_1\cos\phi_1, F(\theta_1, \phi_1)\sin\theta_1\sin\phi_1, F(\theta_1, \phi_1)\cos\theta_1)$$

and write q as

$$(F(\theta_2, \phi_2), F(\theta_2, \phi_2)\sin\theta_2\cos\phi_2, F(\theta_2, \phi_2)\sin\theta_2\sin\phi_2, F(\theta_2, \phi_2)\cos\theta_2)$$

for some $(\theta_i, \phi_i) \in \mathbb{S}^2$, $i = 1, 2$. The fact $p = aq$, for some $a > 0$, then implies $p = q$, which is contradiction. Therefore, Σ is *acausal*.

5. Appendix

In this appendix, we give some Lemmas which are needed to complete the proof of Proposition 3.2. We will follow closely Nirenberg's argument in [17]. First, we introduce some notations: given an integer $k \geq 2$ and a positive number $\alpha < 1$, let

$$\begin{aligned}
\mathcal{E}^{k,\alpha} &= \text{the space of } C^{k,\alpha} \text{ embeddings of } S^2 \text{ into } \mathbb{R}^3 \\
\mathcal{X}^{k,\alpha} &= \text{the space of } C^{k,\alpha}\ \mathbb{R}^3\text{-valued vector functions on } S^2 \\
\mathcal{S}^{k,\alpha} &= \text{the space of } C^{k,\alpha} \text{ symmetric } (0,2) \text{ tensors on } S^2 \\
\mathcal{M}^{k,\alpha} &= \text{the space of } C^{k,\alpha} \text{ Riemannian metrics on } S^2.
\end{aligned}$$

On page 353 in [17], Nirenberg proved

Lemma 5.1. *Let $\sigma \in \mathcal{M}^{4,\alpha}$ be a metric with positive Gaussian curvature. Let $X \in \mathcal{E}^{4,\alpha}$ be an isometric embedding of $(\mathbb{S}^2, \sigma)$ in $\mathbb{R}^3$. There exists two positive numbers ϵ and C, depending only on σ, such that if $\tau \in \mathcal{M}^{2,\alpha}$ satisfying*

$$||\sigma - \tau||_{C^{2,\alpha}} < \epsilon,$$

then there is an isometric embedding $Y \in \mathcal{E}^{2,\alpha}$ of $(\mathbb{S}^2, \tau)$ in $\mathbb{R}^3$ such that

$$||X - Y||_{C^{2,\alpha}} \leq C||\sigma - \tau||_{C^{2,\alpha}}.$$

In what follows, we want to show that the constants ϵ and C in the above Lemma can be chosen to be independent on σ, provided σ is sufficiently close to some $\sigma^0 \in \mathcal{M}^{5,\alpha}$ (see Lemma 5.3). First, we prove the following:

Lemma 5.2. *Let $\sigma^0 \in \mathcal{M}^{5,\alpha}$ be a metric with positive Gaussian curvature. There exists positive numbers δ and $\hat{K}$, depending only on σ^0, such that if $\sigma \in \mathcal{M}^{4,\alpha}$ satisfying*

$$||\sigma^0 - \sigma||_{C^{2,\alpha}} < \delta,$$

then for any $\gamma \in \mathcal{S}^{2,\alpha}$ and any $Z \in \mathcal{X}^{2,\alpha}$, there exists a solution $Y \in \mathcal{X}^{2,\alpha}$ to the linear equation

$$2dX^\sigma \cdot dY = \gamma - (dZ)^2. \tag{5.1}$$

Here $X^\sigma \in \mathcal{E}^{4,\alpha}$ is any given isometric embedding of $(\mathbb{S}^2, \sigma)$. Moreover, for every Z (with γ fixed), a particular solution Y denoted by $\Phi(Z)$ may be chosen so that

$$||\Phi(Z)||_{C^{2,\alpha}} \leq \hat{K}\left(||\gamma||_{C^{2,\alpha}} + ||Z||^2_{C^{2,\alpha}}\right), \tag{5.2}$$

and for any $Z, Z^1 \in \mathcal{E}^{2,\alpha}$,

$$||\Phi(Z) - \Phi(Z_1)||_{C^{2,\alpha}} \leq \hat{K}||Z + Z_1||_{C^{2,\alpha}} \cdot ||Z - Z_1||_{C^{2,\alpha}}. \tag{5.3}$$

Proof. We proceed exactly as in [17]. For any $\sigma \in \mathcal{M}^{4,\alpha}$, let X^σ be a given isometric embedding of $(\mathbb{S}^2, \sigma)$ in $\mathbb{R}^3$. Let X_3 be the unit inner normal to the surface $X^\sigma(\mathbb{S}^2)$. Let $\{u, v\}$ be a local coordinate chart on $\mathbb{S}^2$. Let ϕ, p_1, p_2 be defined as in (6.5), (6.6) in [17]. Then ϕ, p_1, p_2 satisfy the system of equations (6.11)-(6.13) in [17] with c_1, c_2, Δ defined on page 356-357 in [17]. By section 6.3 in [17], the derivatives of Y are completely determined by ϕ, which satisfies (6.15) in [17]. Let ϕ be given by (7.3) in [17], following the first paragraph in section 8.1 in [17], we obtain a unique solution $Y \in \mathcal{E}^{2,\alpha}$ to (5.1), normalized to vanish at a fixed point on $\mathbb{S}^2$. We denote such a Y by $Y = \Phi(Z)$.

To prove estimates (5.2) and (5.3), by the Remark on page 365 in [17] and the proof following it, we know it suffices to show

$$||Y||_{C^{2,\alpha}} \leq C\left[||d\bar{\sigma}^2||_{C^{1,\alpha}} + ||\frac{1}{\Delta}(c_{1v} - c_{2u})||_{C^{\alpha}}\right], \tag{5.4}$$

where $d\bar{\sigma}^2 = \gamma - (dZ)^2$, c_{1v}, c_{2u} are derivatives of c_1, c_2 with respect to v, u respectively. On the other hand, by section 9 in [17], to prove (5.4), it suffices to establish an $C^{1,\alpha}$ estimate of ϕ:

$$||\phi||_{C^{1,\alpha}} \leq C||d\bar{\sigma}^2||_{C^{1,\alpha}}. \tag{5.5}$$

Therefore, in what follows, we will prove that there are positive numbers δ and C, depending only on σ^0, such that (5.5) holds for any σ satisfying $||\sigma^0 - \sigma||_{C^{2,\alpha}} < \delta$.

We first recall the fact that ϕ is a solution to the second order elliptic equation (6.15) in [17]. For simplicity, we let

$$L_\sigma(\phi) = \mathcal{L}(\phi_u, \phi_v), \quad F_\sigma(d\bar{\sigma}^2) = \mathcal{L}(c_1, c_2) - T, \tag{5.6}$$

where $\mathcal{L}(\phi_u, \phi_v)$, $\mathcal{L}(c_1, c_2) - T$ are given as in (6.16) and (6.14) in [17], then (6.15) in [17] becomes

$$L_\sigma(\phi) + H_\sigma\phi = F_\sigma(d\bar{\sigma}^2), \tag{5.7}$$

where H_σ is the mean curvature of $X^\sigma(\mathbb{S}^2)$ w.r.t X_3 (note that our H here equals $2H$ in [17]). On the other hand, since we have chosen ϕ to be given by the integral formula (7.3) in [17], we know ϕ is a special solution to (5.7) in the sense that ϕ is L^2-perpendicular to the kernel of the operator $L_\sigma(\cdot) + H_\sigma$ (See page 359 in [17]). For any $\sigma \in \mathcal{M}^{4,\alpha}$, let $Ker(\sigma)$ denote the space of solutions ψ to the the homogeneous equation

$$L_\sigma(\psi) + H_\sigma\psi = 0. \tag{5.8}$$

On page 360 in [17], it was shown that $Ker(\sigma)$ is spanned by the coordinate functions of X_3.

Note that the coefficient of (5.7) depends only on the metric σ. Therefore, if σ is close to σ^0 in $C^{2,\alpha}$, we know by Theorem 8.32 in [10] that, to prove the $C^{1,\alpha}$ estimate (5.5), it suffices to prove the following C^0 estimate

$$||\phi||_{C^0} \leq C||d\bar{\sigma}^2||_{C^{1,\alpha}}, \tag{5.9}$$

where C is some positive constant independent on σ, provided σ is sufficiently close to σ^0 in $C^{2,\alpha}$.

Suppose (5.9) is not true, then there exists $\{\sigma_i\} \subset \mathcal{M}^{4,\alpha}$ which converges to σ^0 in $C^{2,\alpha}$, $\{d\bar{\sigma}_i^2\} \subset \mathcal{S}^{1,\alpha}$ with $||d\bar{\sigma}_i^2||_{C^{1,\alpha}} = 1$, and a sequence

of numbers $\{C_i\}$ approaching $+\infty$ so that the corresponding ϕ_i (of $Y = Y_i$) satisfies

$$||\phi_i||_{C^0} \geq C_i.$$

Consider $\xi_i = \phi_i/||\phi_i||_{C^0}$, then ξ_i satisfies

$$L_{\sigma_i}(\xi_i) + 2H_{\sigma_i}\xi_i = ||\phi_i||_{C^0}^{-1} F_{\sigma_i}(d\bar{\sigma}_i^2). \tag{5.10}$$

By Theorem 8.32 in [10], we conclude from (5.10) and the facts $\{\sigma_i\}$ converges to σ^0 in $C^{2,\alpha}$ and $||\xi_i||_{C^0} = 1$ that

$$||\xi_i||_{C^{1,\alpha}} \leq C, \tag{5.11}$$

where C is some positive constant independent on i. Now (5.11) implies that ξ_i converges in C^1 to some ξ which is also in $C^{1,\alpha}$. Moreover, $||\xi||_{C^0} = 1$. By (5.10), ξ is a weak solution to the equation

$$L_{\sigma^0}\xi + H_{\sigma^0}\xi = 0. \tag{5.12}$$

Since $\sigma^0 \in C^{5,\alpha}$, the coefficients of (5.12) (given by (6.16) in [17]) are then in $C^{3,\alpha}$, hence in $C^{2,1}$. By Theorem 8.10 in [10], we know $\xi \in W^{4,2}$, hence in C^2. Therefore, ξ is a classic solution to (5.12), i.e. $\xi \in Ker(\sigma^0)$. On the other hand, we know ϕ_i, hence ξ_i, is L^2-perpendicular to $Ker(\sigma_i)$ for each i. Since $\{\sigma_i\}$ converges to σ^0 in $C^{2,\alpha}$ and $\{\xi_i\}$ converges to ξ in C^1, we conclude that ξ must be L^2-perpendicular to $Ker(\sigma^0)$. Hence, ξ must be zero. This is a contradiction to the fact $||\xi||_{C^0} = 1$. Therefore, we conclude that (5.9) holds.

As mentioned earlier, once we establish the C^0 estimate (5.9), we will have the $C^{1,\alpha}$ estimate (5.5). Then we can proceed as in the rest of section 9 in [17] to prove (5.4), hence prove (5.2) and (5.3). □

We note that the constants ϵ and C in Lemma 5.1 indeed can be chosen as $\epsilon = \frac{1}{4\bar{K}^2}$ and $C = 2\bar{K}$, where $\bar{K}$ is the constant in Theorem 2' on page 352 in [17]. Therefore, by applying the exactly same iteration argument as on page 352-353 in [17], one concludes from Lemma 5.2 that

Lemma 5.3. *Let $\sigma^0 \in \mathcal{M}^{5,\alpha}$ be a metric with positive Gaussian curvature. There exists positive numbers δ, ϵ and C, depending only on σ^0, such that for any $\sigma \in \mathcal{M}^{4,\alpha}$ satisfying*

$$||\sigma^0 - \sigma||_{C^{2,\alpha}} < \delta,$$

if $\tau \in \mathcal{M}^{2,\alpha}$ satisfying

$$||\sigma - \tau||_{C^{2,\alpha}} < \epsilon,$$

then there is an isometric embedding $Y \in \mathcal{E}^{2,\alpha}$ of $(\mathbb{S}^2, \tau)$ in $\mathbb{R}^3$ such that

$$||X - Y||_{C^{2,\alpha}} \leq C||\sigma - \tau||_{C^{2,\alpha}}.$$

Here $X \in \mathcal{E}^{4,\alpha}$ *is any given isometric embedding of* $(\mathbb{S}^2, \sigma)$.

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School of Mathematical Sciences, Monash University, Victoria, 3800, Australia.
E-mail address: Pengzi.Miao@sci.monash.edu.au

Key Laboratory of Pure and Applied mathematics, School of Mathematics Science, Peking University, Beijing, 100871, P.R. China.
E-mail address: ygshi@math.pku.edu.cn

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.
E-mail address: lftam@math.cuhk.edu.hk